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USING BLENDED LEARNING TO SUPPORT STUDENTS THINKING POWERS IN MULTIVARIABLE CALCULUS

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Abstract. Promoting mathematical thinking plays a significant role in the learning of advanced mathematics. Researchers usually try to support the students to overcome their difficulties in multivariable calculus by promoting mathematical thinking. The literature review and our preliminary study reveal that there are still certain difficulties faced by students when they adopt the new mathematical ideas and objects in multivariable calculus through mathematical thinking approach. In this paper, the theoretical framework for promoting mathematical thinking by computer has been discussed and theoretical reasons for selecting blended learning for promoting mathematical thinking in multivariable calculus have been put forward. Blended learning by integration of the benefits of both face-to-face and computer environments have been suggested as a relevant environment to help students in the learning of multivariable calculus through mathematical thinking approach.

Keywords: Blended learning; Mathematical thinking; multivariable calculus; students' difficulties

Abstrak. Dalam pembelajaran di peringkat tinggi, pemikiran matematik boleh memainkan peranan yang signifikan. Dalam kalangan penyelidik terdapat di antara mereka yang mempromosikan pemikiran matematik untuk menyokong pembelajaran yang boleh meringankan kepayahan dalam kalkulus multi pemboleh ubah. Namun begitu, kajian literatur dan kajian awal yang telah dijalankan menunjukkan masih wujud kepayahan tertentu apabila mereka mengamalkan pendekatan pemikiran mnatematik. Dalam artikel ini, rangka kerja teori mempromosikan pemikiran matematik berkomputer dibincangkan bersama rasional memilih *blended learning* untuk mempromosikan pemikiran matematik dalam Kalkulus Multi Pembolehubah. Seterusnya dibincangkan kesesuaian *blended learning yang* mengintegrasikan suasana pembelajaran bersemuka dan berkomputer untuk menyokong pembelajaran pemikiran matematik.

Kata kunci: Blended learning; pemikiran Matematik; kalkulus multi pemboleh ubah; kesukaran pelajar

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1.0 INTRODUCTION

Mathematical thinking is the main goal of mathematics education (Kardage, 2008) and it can play an important role as a way of learning and teaching mathematics (Stacy, 2006). There are quite a number of extensive studies done on mathematical thinking such as works by Mason, Burton & Stacey (1982), Dubinsky (1991), Schoenfeld (1992), Yudariah & Tall (1999), Gray & Tall (2001), Tall (2004), and Roselainy (2009). Tall (2004) categorized mathematical thinking into three significant worlds: *conceptual embodied* world, *proceptual symbolic* world, and *axiomatic formal* world. The theory of three worlds of mathematical thinking provides an appropriate and rich structure to understand and interpret mathematical learning and thinking at all levels particularly in undergraduate mathematics (Tall, 2007).

Mathematics is a prime constituent and infrastructure of the education of undergraduate students in many fields. Calculus, particularly multivariable calculus, is one of the most important parts of mathematics syllabus for undergraduate students. It is offered as a prerequisite course to other advanced mathematics courses and even to other courses. There are some conflicts in the learning of calculus that makes it to be seen as one of the most difficult courses to study (Tall & Schwarzenberger, 1978; Tall, 1993a; Kashefi, Zaleha & Yudariah, 2010a, 2011). Researches try to support students in the learning of calculus and help them to overcome their difficulties by promoting three worlds of mathematical thinking with or without using computers.

Researchers like Tall tried to help students in the learning of mathematics by using computers. Tall in many researches (1986, 1989, 1990, 1993b, 1998, 2003) used a computer software to support students' mathematical thinking and to help them overcome their difficulties in the learning of calculus based on Socratic dialogue between teacher and students which is enhanced by the addition of the computer facilities such as visualization tools. Tall, in these researches focused more on the concepts of basic calculus.

In a study on multivariable calculus, (Roselainy, 2009; Roselainy, Yudariah & Mason, 2007; Roselainy, Sabariah & Yudariah, 2007; and Sabariah, Yudariah & Roselainy, 2008) designed a model of active learning based on mathematical thinking approach to support the teaching and learning of multivariable calculus without using computer. They focused on three major aspects of teaching and learning: the development of mathematical knowledge construction, mathematical thinking processes, and generic skills (Kashefi, Zaleha & Yudariah, 2010a). Roselainy and her colleagues used themes and mathematical processes through specially designed prompts and questions to invoke and support the students in using their own mathematical thinking powers during face-to-face (F2F) interactions in a classroom setting.

However, a previous study, which implemented the Roselainy *et al.*'s method, found that students still have difficulties when encountering non-routine problems in multivariable calculus. Literature review indicates that there are very little researches in promoting mathematical thinking by using computers.

The purpose of this paper is to provide a theoretical framework that supports blended learning as a relevant environment to promote mathematical thinking. A theoretical framework that supports blended learning by the integration of both F2F and computer environments has a rich structure to support students to overcome their difficulties in their learning of multivariable calculus.

2.0 MATHEMATICAL THINKING

Mathematical thinking is a dynamic process which expands our understanding with highly complex activities, such as abstracting, specializing, conjecturing, generalizing, reasoning, convincing, deducting, and inducting (Mason, Burton & Stacey, 1982; Tall, 1991; Yudariah & Roselainy, 2004). Mathematical thinking as a main segment of mathematics education can play an important role in the learning and teaching of mathematics.

Based on the theory of three modes of representation of human knowledge (Bruner, 1966), enactive, iconic and symbolic are the three forms of representation in mathematics. Tall (1995) noted that the various forms of symbolic representation are: verbal (language, description), formal (logic, definition), and proceptual (numeric, algebraic etc). In further studies, Tall (2004, 2007), based on Bruner's theory, stated that there are not only three distinct types of mathematics worlds; in fact there are actually three significantly different worlds of mathematical thinking as follows (see Figure 1).

- the *conceptual embodied* world of our physical perceptions and actions in a real-world context that we build into mental conceptions through reflection on objects,

- the *proceptual symbolic* world that begins with real-world actions (e.g. differentiation, integration) and symbolization into concepts (e.g. derivative, integral), developing symbols that operate both as processes to do and concepts to think about (called procept),
- the *axiomatic formal* world of axiomatic systems based on formal definitions and proof (e.g. group, field).

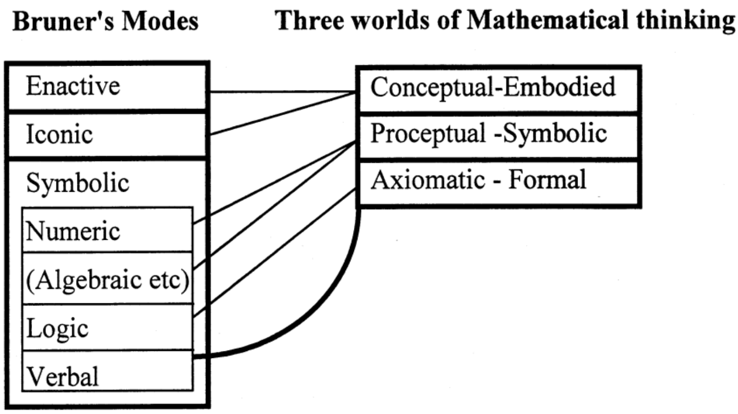

Figure 1 The relation between three Bruner's modes and three worlds of mathematical thinking

The three distinct types of mathematical thinking as *embodied*, *symbolic* and *formal* are also particularly appropriate in the calculus (Tall, 2007). Calculus, as an important course for undergraduate students, requires them to work with several mathematical ideas and various representations and also to use this knowledge in their fields of study (Roselainy, Sabariah & Yudariah, 2007). However, many students have difficulties when they encounter the new mathematics ideas and non-routine questions in calculus. In the next two sub-sections, we will explain how researchers try to help students to overcome their difficulties in calculus by promoting mathematical thinking. Then, we will show how much these methods are capable of supporting students' ability to overcome their difficulties and which difficulties still exist.

2.1 Promoting Mathematical Thinking in F2F Classroom

In the F2F learning environment, the *"didactic triangle"* represents a model with the multiple relations among the three vertices: the student, the teacher, and the mathematics (Tall, 1989; Albano, 2005). Therefore, each model and strategy that is used to promote mathematical thinking must identify the role of each components and relations among them.

The earlier study of multivariable calculus by Roselainy and her colleagues presented a model of active learning that they had developed and implemented in the teaching of multivariable calculus in UTM. Their model is based on invoking students' mathematical thinking powers, supporting mathematical knowledge construction, and promoting generic skills. They had use themes and mathematical processes through specially designed prompts and questions to invoke and support students to use their own mathematical thinking powers during F2F interactions in a classroom setting. In other words, they had provided and promoted a learning environment where the mathematical powers were used specifically and explicitly, towards supporting the students (i) to become more aware of the mathematics structures being learned, (ii) to recognize and use their mathematical thinking powers, and (iii) to modify their mathematical learning behavior (Kashefi, Zaleha & Yudaria, 2010a).

However, many students still struggle as they encounter the new mathematical ideas and objects in multivariable calculus, especially the functions of two variables (Kashefi, Zlaleha & Yudariah, 2010a, 2011). Some student difficulties are: (i) sketching the graph of two-variable functions in 3-dimensions, (ii) confusion in the use of symbols in representing two-variable functions, (iii) students' idiosyncrasies attributed from previous mathematical construction, (iv) inability to select appropriate representation of mathematics worlds, (v) transition from one world to other world of mathematics.

For instance, some students found the range of: $f(x,y) = \sqrt{64 - 4x^2 - y^2}$ as shown in a typical student's response in Figure 2. This student wrote the range of f as $R_f = \{z| -8 \leq z \leq 8\}$ based on the graph of the domain. This difficulty might be related to the negative effect of students' previous knowledge in finding the range of single-variable function based on the graph. In fact, the previous construction of the range of single-variable function was recalled in order to find the range of the two-variable function. Thus, the student found the range of two-variable function in a wrong way based on the values of *y* on the *y*-axis for the graph of the domain.

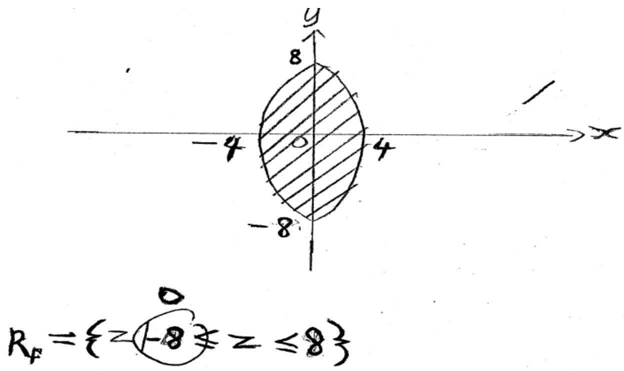

Figure 2 A typical student's mistake in finding the range of $f(x,y) = \sqrt{64 - 4x^2 - y^2}$

In another example, in finding the domain of $f(y,z) = 9 - y^2 - z^2$, some students wrote the domain in terms of *x* and *y* as $D_f = \{(x,y)|\ x, y \in R\ \}$. Most of these students noted that they tend to confuse this problem with $f(y,z) = 9 - x^2 - y^2$. Figure 3 shows a typical student's response in finding and sketching the domain of *f*. The student had not only incorrectly identified the domain as $\{(x,y)|\ x, y \in R\ \}$ but also sketched it in the wrong coordinate plane. The student was not aware of the different symbols used and their role in representing the function. The obstacle faced was due to their inflexibility in handling symbolic representation.

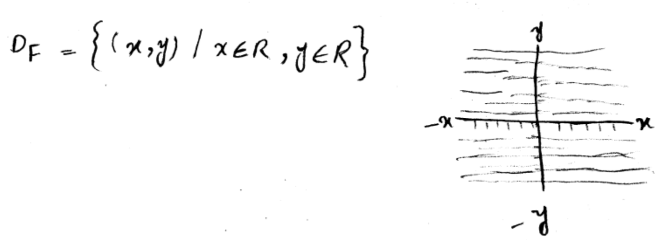

Figure 3 A typical student's attempt in finding the domain of $f(y,z) = 9 - y^2 - z^2$

Sketching the graphs was the main difficulty among students in solving problems in multivariable calculus. For example, some students had difficulties in finding the volume of the solid cut of the sphere $x^2 + y^2 + z^2 = 4$ by the cylinder $x^2 + y^2 = 2y$. Identifying the bounded solid between surfaces and finding the limits of integration were the common difficulties among students. The lack of mastery of the integration techniques was another reason of difficulties for a few students. Figure 4 shows a typical student's response in which the student sketched the graphs incorrectly.

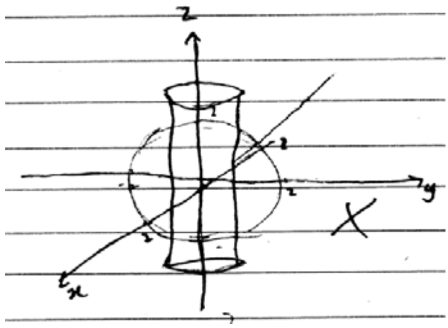

Figure 4 Atypical student's attempt in sketching the graphs

Poor prior knowledge or even lack of it was another difficulty that student showed in the learning of multivariable calculus. Some students did not posses enough prior knowledge and background to solve the problems. For instance, Figure 5 represents a typical student's response that the student found the multiply of $\sin x^2$ to x as $\sin x^3$.

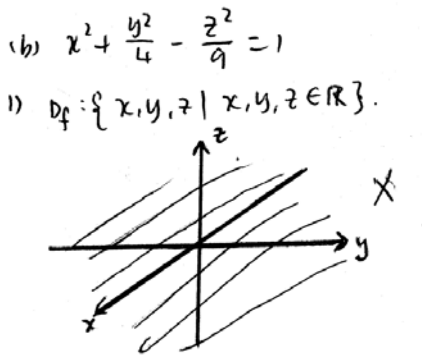

Figure 5 A student's mistake in algebraic manipulation

2.2 Promoting Mathematical Thinking with Computer

The theory of Skemp (1979) identifies three modes of building and testing of conceptual structures as shown in Table 1. In the process of mathematical knowledge construction, one, two, or three modes of reality building can be used in combination with one, two, or three modes of reality testing.

Table 1 Reality construction

Mode	Reality Building	Reality testing
1	experience	experiment
2	communication	discussion
3	creativity	internal consistency

According to Skemp, pure mathematics relies on Mode 2 and 3, but it is not at all based only on Mode 1 (Tall, 1986). Tall showed that how computer environment brings a new refinement to the theory of Skemp and extended Skemp's theory to four modes: Inanimate, Cybernetic, Interpersonal, and Personal. The last of these corresponds to Skemp's Mode 3. The interpersonal mode of building and testing concept also corresponds to Skemp's Mode 2, whilst the first two are a modification of Skemp's Mode 1 (Tall, 1989, 1993b). In fact, the computer provides an environment that gives a new way for building and testing mathematical concept by supporting all modes. Therefore, computer environment can be used in all of these modes and learners can also build mathematical concepts by considering the examples (and non-examples) of process in interaction with this environment especially in the embodied world of mathematics (Tall, 1986).

In other words, computer environment provides not only a numeric computation and graphical representation; it also allows manipulation of objects by an enactive interface (Tall, 1986) that by using them, we can support the students' knowledge construction and help them to overcome their difficulties in the embodied world of mathematics. Tall (1989), by combination of a human teacher as guide (organizing agent) and a computer environment (generic organiser) for teaching, tried to support the students' mathematical thinking (see Figure 6). In

Tall's method, teachers as organizing agent do not have a directive role but they can only answer questions which may arise in the course of the student investigations through a Socratic dialogue (Skemp's Mode 2) which is then enhanced by the presence of computers (Tall, 1986, 2004).

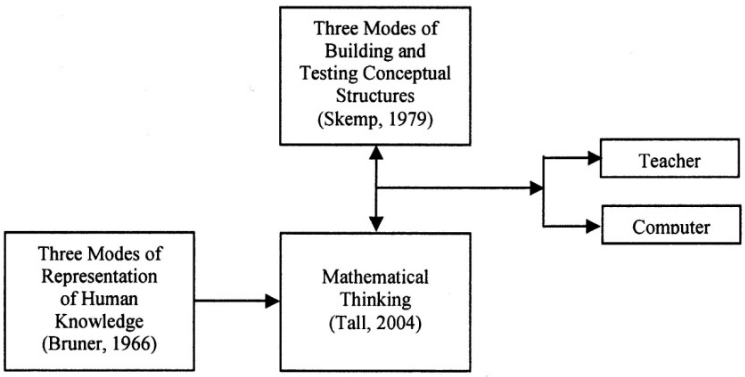

Figure 6 The relation between the theories of Bruner, Tall, and Skemp

3.0 BLENDED E-LEARNING

Blended learning as an environment in benefiting both F2F and online environments is another new trend in the teaching and learning of mathematics. However, just like so many terms within this field, it remains ill-defined and there is still quite a bit of ambiguity about what it means (Oliver & Trigwell, 2005; Graham, 2006; Hisham Dzakiria et al., 2006). There are many definitions (Oliver & Trigwell, 2005; Graham, 2006; Huang, Ma & Zhang, 2008) for blended learning. The three common definitions are: (i) The combination of instructional delivery media (Orey, 2002), (ii) The combination of instructional methods (Driscoll, 2002), and (iii) The combination of online and F2F instruction (Reay, 2001). In the third definition, blended learning is seen as a fusion of learning concepts which integrates traditional F2F sessions with the e-learning elements (Reay, 2001) in order to obtain the benefits of both learning forms.

In this paper, we defined blended learning as the combination of F2F formats and e-learning formats (Reey, 2001) that identified the environment as having two important components of Tall's method: generic organizer (computer) and organizing agent (teacher) (Kashefi, Zaleha & Yudariah, 2010b). In fact, blended learning environment is rich with tools to extend Tall's approach in promoting mathematical thinking in multivariable calculus. Furthermore, the use of e-learning as an important element of blended learning has appropriate tools to support students' generic skills. It is proposed that blended learning has the potential to improve Roselainy et al.'s model for supporting students in three aspects of learning.

On the other hand, Fahlberg-Stojanovska & Stojanovski (2007) noted that the best learning can take place when all three primary senses of seeing (visual), hearing (audio) and doing (enactive) were involved in an interactive environment. They proposed links between these senses and the two components of blended learning as shown in the following figure (see Figure 7).

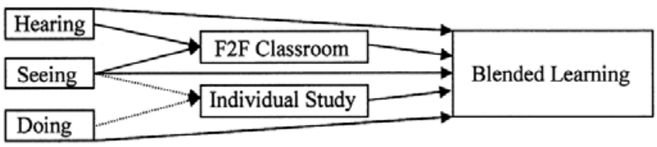

Figure 7 The relation between three primary senses and blended learning

Moreover, based on repeated studies, Muir (2001) reported that students learned in different ways such as reading, hearing, seeing and doing, but the best learning occurred when students learned through the combination of these senses. Thus, blended learning has the potential to involve all these senses effectively compared to using computer or lecture separately. Therefore, due to the relation between Bruner's modes and primary senses on one hand, and also the relation between primary senses and blended learning on the other hand, we can see a link between Bruner's theory and the components of blended learning (Kashefi, Zaleha & Yudariah, 2010b). See Figure 8.

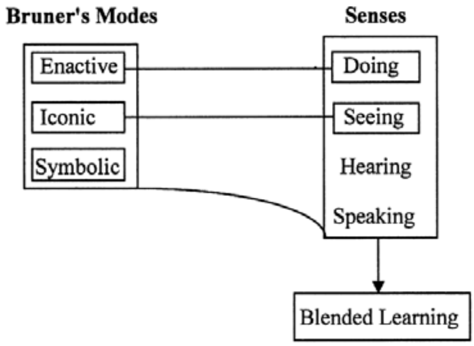

Figure 8 The relation between three Bruner's modes and blended learning through primary senses

4.0 DISCUSSION AND CONCLUSION

In this research article, we have reviewed the earlier studies (Yudariah & Roselainy, 2004; Roselainy, Sabariah & Yudariah, 2007; and Sabariah, Yudariah & Roselainy, 2008) in F2F multivariable calculus. It is found that there is an effective way to provide opportunities for interactions between lecturer, students, and mathematics to promote mathematical thinking. Furthermore, students' difficulties in the learning of multivariable calculus indicates that Roselainy et al.'s method can help in making the mathematical thinking processes an explicit learning. It also highlights the students' struggle as they encounter new mathematical ideas and concepts.

We have also explained the theory of the three worlds of mathematical thinking by Tall based on Bruner's theory. Moreover, the theoretical framework for promoting mathematical thinking by using computer has also been discussed. We have explained how Tall in many researches (1986, 1989, 1990, 1993b, 1998, 2003) tried to support students' thinking powers in calculus by using generic organiser (as programmed on a computer) and organizing agent (that can be a teacher) based on Skemp's theory. The relationship among Bruner's theory, Skemp's theory, Tall's theory and the two important components of promoting mathematical thinking was shown in Figure 6.

Theoretical reasons have supported that blended learning, by the integration of the benefits of both F2F multivariable calculus classroom and computer environment, can be selected as an environment to support students' mathematical thinking powers. On the other hand, blended learning by involving all primary senses makes a strong link between its elements and Bruner's theory. Figure 9 shows the relation between mathematical thinking and blended learning through Bruner and Skemp theories.

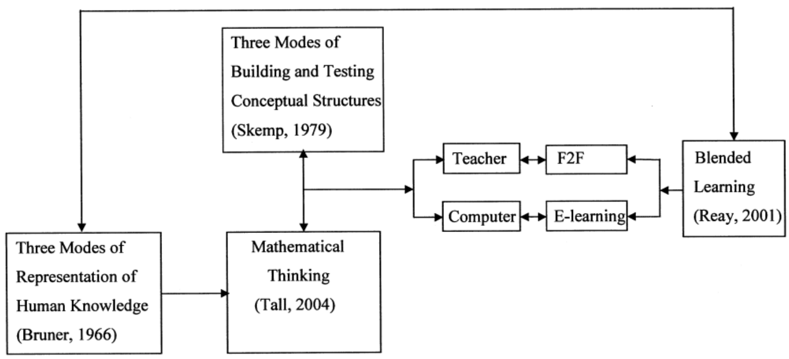

Figure 9 The relation between mmathematical thinking and blended learning

As a conclusion, findings of this study proposed blended learning as a sufficient environment to support students' thinking powers in multivariable calculus. The results obtained from this study are expected to be useful in designing activities and tools to teach multivariable calculus based on mathematical thinking. It is also hoped that these activities and tools will support students to overcome their difficulties in this course.

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